Rheinische Friedrich-Wilhelms-Universität Bonn
Institut für Informatik I

Tom Kamphans Elmar Langetepe

Optimal competitive online ray search with an error-prone robot

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Abstract

We consider the problem of finding a door along a wall with a blind robot that neither knows the distance to the door nor the direction towards of the door. This problem can be solved with the well-known doubling strategy yielding an optimal competitive factor of 9 with the assumption that the robot does not make any errors during its movements. We study the case that the robot's movement is erroneous. In this case the doubling strategy is no longer optimal. We present optimal competitive strategies that take the error assumption into account. The analysis technique can be applied to different error models.

Keywords: Online algorithm, Online motion planning, competitive analysis, ray search, errors.

1 Introduction

Motion planning in unknown environments is theoretically well-understood and also practically solved in many settings. During the last decade many different objectives where discussed under several robot models. For a general overview of theoretical online motion planning problems and its analysis see the surveys [4, 22, 13, 24, 24]

Theoretical correctness results and performance guarantees often suffer from idealistic assumptions so that in the worst case a correct implementation is impossible. On the other hand practioners analyze correctness results and performance guarantees mainly statistically or empirically. Therefore it is useful to investigate, how theoretic online algorithms with idealistic assumptions behave, if those assumptions cannot be fulfilled. More precisely, can we incorporate assumptions of errors in sensors and motion directly into the theoretical sound analysis? We already successfully considered the behaviour of the well-known pledge algorithm, see e. g. Abelson and diSessa [1], and Hemmerling [11], in the presence of errors [16]. It has been proven that a compass with an error of $\pi/2$ is sufficient to leave an unknown maze with a strategy that makes use of counting turning angles.

The task of finding a point on a line by a blind agent without knowing the location of the goal was considered by Gal [7, 8, 2] and independently reconsidered by Baeza-Yates et al. [3]. Both approaches lead to the so called doubling strategy, which is a basic paradigma for searching algorithms, e. g. searching for a point on m rays, see [7], or approximating the optimal search path, see [6].

Searching on the line was generalized to searching on m concurrent rays starting from a single source, see [3, 7]. Many other variants were discussed since then, for example m-ray searching with restricted distance (Hipke et al. [12], Langetepe [20], Schuierer [21]), m-ray searching with additional turn costs (Demaine et al. [5]), parallel m-ray searching (Hammar et al. [10]) or

randomized searching (Kao et al. [18]). Furthermore, some of the problems were again rediscovered by Jaillet et al. [14].

In this paper we investigate how an error in the movement influences the correctness and the corresponding competitive factor of a strategy. The error range, denoted by a parameter δ, may be known or unknown to the strategy.

The paper is organized as follows. In Section 2 we recapitulate some details on m-ray searching and its analysis. The error model is introduced in Section 2.1. In Section 3 we discuss the case where the strategy is not aware of errors, therefore we analyze the standard doubling strategy showing correctness and performance results. The main result is presented in Section 4. We can prove that the optimal competitive strategy that searches for a goal on a line achieves a factor of $1 + 8\left(\frac{1+\delta}{1-\delta}\right)^2$ if the error range δ is known. Fortunately, our analysis technique works for different error models and is generic in this sense. Finally, we consider the m-ray searching in Section 5. For a summary of the results and factors see Section 6. A preliminary version of this report appeared in [17].

2 The standard problem

The task is to find a door in a wall, respectively a point, t, on a line. The robot does not know whether t is located left hand or right hand to its start position, s, nor does it know the distance from s to t. Baeza-Yates et al. [3] describe a strategy for solving this problem by using a function f. $f(i)$ denotes the distance the robot walks in the i-th iteration. If i is even, the robot moves $f(i)$ steps from the start to the right and $f(i)$ steps back; if i is odd, the robot moves to the left. It is assumed that the movement is correct, so after moving $f(i)$ steps from the start point to the right and moving $f(i)$ steps to the left, the robot has reached the start point. Note, that this does not hold, if there are errors in the movement, see for example Figure 2.

The competitive analysis compares the cost of a strategy to the cost of an optimal strategy that knows the whole environment. In our case these cost is given by the distance, d, to the goal. With the assumption $d \geq 1$ a search strategy that generates a path of length $|\pi_{\text{onl}}|$ is called *C-competitive* if for all possible scenarios $\frac{|\pi_{\text{onl}}|}{d} \leq C$ holds[1]. It was shown by Baeza-Yates et al. [3] that the strategy $f(i) = 2^i$ yields a competitive factor of 9 and that no other strategy will be able to achieve a smaller factor.

The problem was extended to m concurrent rays. It was shown by Gal [8] that w.l.o.g. a strategy visits the rays in a cyclic order and with increasing distances $f(i) < f(i+1)$. The optimal competitive factor is given by $1 +$

[1]For $d \geq 1$ the constant that usually appears in the definition of the competitive factor can be omitted see for example [20].

$2\frac{m^m}{(m-1)^{m-1}}$ and an optimal strategy is defined by $f(i) = \left(\frac{m}{m-1}\right)^i$, see [3, 7]

2.1 Modelling the error

The robot moves straight line segments of a certain length from the start point alternately to the left and to the right. Every movement can be errorneous, which causes the robot to move more or less far than expected. However, we require that the robots error per unit is within a certain error bound, δ. More precisely, let f denote the length of a movement required by the strategy—the nominal value— and let ℓ denote the actually covered distance, then we require that $\ell \in [\,(1-\delta)f, (1+\delta)f]$ holds for $\delta \in [\,0, 1[$, i. e. the robot moves at least $(1-\delta)f$ and at most $(1+\delta)f$. This is a reasonable error model, since the actually covered distance is in a symmetrical range around the nominal value. Another commonly used method is to require $\ell \in [\,\frac{1}{1+\delta'}f, (1+\delta')f]$ for $\delta' > 0$. This leads to an unsymmetrical range around the desired value, but does not restrict the upper bound for the error range. Since both error models may be of practical interest, we give results for both models, although we give full proofs only for the first model. We call the first model *percentual error*, the second model *standard multiplicative error*.

3 Disregarding the error

In this section, we assume that the robot is not aware of making any errors. Thus, the optimal doubling strategy presented above seems to be the best choice for the robot. In the following we will analyze the success and worst case efficiency of this strategy with respect to the unknown δ.

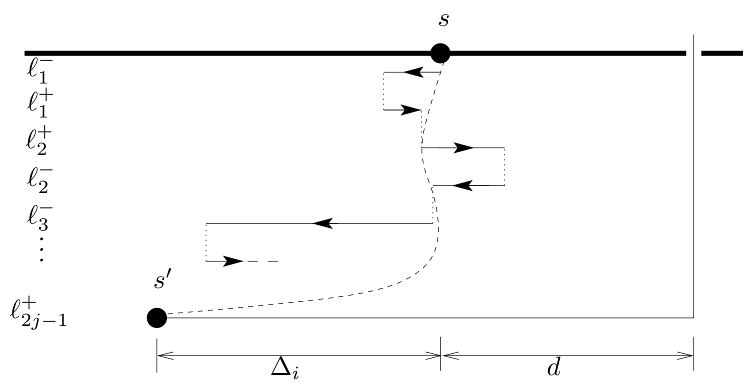

Figure 1: The i-th iteration consists of two separate movements, ℓ_i^+ and ℓ_i^-. Both may be of different length, causing a drift. The vertical path segments are to highlight the single iterations, the robot moves on horizontal segments only.

Since the errors in the movements away from the door and back towards

the door may be different, the robot may not return to the start point, s, between two interations, see Figure 1. Even worse, the start point of every iteration may continuously drift away from the original start point. Let ℓ_i^+ be the length of the movement to the right in the i-th step and ℓ_i^- be the covered distance to the left. Now, the deviation from the start point after the k-th iteration step, the *drift* Δ_k, is

$$\Delta_k = \sum_{i=1}^{k} (\ell_i^- - \ell_i^+)\,.$$

If the drift is greater than zero, the start point s_{k+1} of the iteration $k+1$ is located left to the original start point, if it is smaller than zero, s_{k+1} is right hand to s. Note, that ℓ_i^+ equals ℓ_i^- in the error-free case. We will show that the worst case is achieved, if the robot's drift to the left is maximal. The length of the path π_k after k iterations is

$$|\pi_k| = \sum_{i=1}^{k} (\ell_i^- + \ell_i^+)\,.$$

Theorem 1 *In the percentual error model $[(1-\delta)f, (1+\delta)f]$ with $\delta \in [0, 1[$ the robot will find the door with the doubling strategy $f(i) = 2^i$, if the error δ is not greater than $\frac{1}{3}$. The generated path is never longer than*

$$8\frac{1+\delta}{1-3\delta} + 1$$

times the shortest path to the door[2].

Proof. We assume that finally the goal is found on the right side. The other case is handled analogously. For the competitive setting it is the worst, if the door is hit in the iteration step $2j + 2$ to the right side, but located just a little bit further away than the rightmost point that was reached in the preceeding iteration step $2j$. In other words, another full iteration to the left has to be done and the shortest distance to the goal is minimal in the current situation. Additionally we have to consider the case where the goal is exactly one step away from the start. We discuss this case at the end.
We want the door to be located closely behind the rightmost point visited in the iteration step $2j$. Considering the drift Δ_{2j-1}, the distance from the start point s to the door is

$$d = \ell_{2j}^+ - \sum_{i=1}^{2j-1} (\ell_i^- - \ell_i^+) + \varepsilon\,.$$

[2]More precisely, the factor is $1 + 8\frac{1+\delta}{1-3\delta+\varepsilon}$ for an arbitrary small ε, which is crucial for the case $\delta = \frac{1}{3}$, but neglectable in all other cases. For convenience we omit the ε in this and the following theorems.

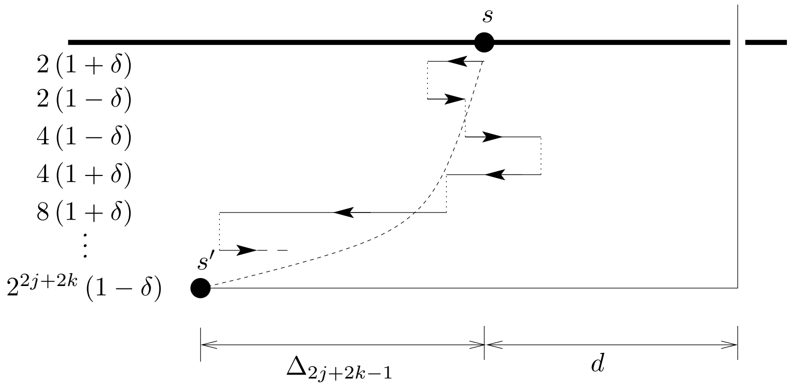

Figure 2: In the worst case, the start point of every iteration drifts away from the door.

The total path length is the sum of the covered distances up to the start point of the last iteration, the distance from this point to the original start point s (i. e. the overall drift), and the distance to the door:

$$|\pi_{\text{onl}}| = \sum_{i=1}^{2j+1} (\ell_i^- + \ell_i^+) + \sum_{i=1}^{2j+1} (\ell_i^- - \ell_i^+) + d.$$

With this we get the worst case ratio

$$\frac{|\pi_{\text{onl}}|}{d} = 1 + \frac{\sum_{i=1}^{2j+1}(2\ell_i^-)}{\ell_{2j}^+ - \sum_{i=1}^{2j-1}(\ell_i^- - \ell_i^+) + \varepsilon}\,. \tag{1}$$

We can see that this ratio achieves its maximum if we maximize every ℓ_i^-, i. e. if we set ℓ_i^- to $(1+\delta)\,2^i$ in this error model. Now we only have to fix ℓ_i^+ in order to maximize the ratio. Obviously, the denominator gets its smallest value if every ℓ_i^+ is as small as possible, therefore we set ℓ_i^+ to $(1-\delta)\,2^i$. So the worst case is achieved if every step to the right is too short and every step to left is too long, yielding a maximal drift to the left and away from the door, see Figure 2[3]. Altogether, we get

$$\begin{aligned}
\frac{|\pi_{\text{onl}}|}{d} &= 1+\frac{\sum_{i=1}^{2j+1}(2\ell_i^-)}{\ell_{2j}^+ - \sum_{i=1}^{2j-1}(\ell_i^- - \ell_i^+) + \varepsilon} = 1+\frac{2\,(1+\delta)\,\sum_{i=1}^{2j+1} 2^i}{(1-\delta)\,2^{2j} - 2\delta\sum_{i=1}^{2j-1} 2^i + \varepsilon} \\
&= 1 + 2\,\frac{(1+\delta)(2^{2j+2}-2)}{(1-3\delta)\,2^{2j} + 4\delta + \varepsilon} \qquad (2) \\
&< 1 + 8\,\frac{1+\delta}{1-3\delta}\,. \qquad (3)
\end{aligned}$$

[3]The vertical path segments are to highlight the single iteration steps, the robot moves only on horizontal segments.

For the case that the goal is exactly one step away from the start we achieve the worst case factor $1 + 4\frac{(1+\delta)}{1}$ which is smaller than the above worst case.

Obviously, we have to require that $\delta \leq \frac{1}{3}$ holds. Otherwise, in the worst case the distance $(1 - 3\delta)\, 2^{2j} + 4\delta$ from the start point does not exceed the point 4δ and we will not hit any goal farther away. □

Proposition 1 *In the standard multiplicative error model* $[\frac{1}{(1+\delta)} f, (1+\delta) f]$ *for* $\delta > 0$*, the doubling strategy always finds the goal with a competitive factor of* $1 + 8\, \frac{(1+\delta)^2}{2-(1+\delta)^2}$ *if* $\delta \leq \sqrt{2} - 1$ *holds.*

Proof. We exactly follow the proof of Theorem 1. The worst case ratio is given by Equation 1 and now we maximize this value by $\ell_i^- = 2^i(1+\delta)$ and $\ell_i^+ = \frac{2^i}{(1+\delta)}$. Using these settings in (3) yields

$$
\begin{aligned}
\frac{|\pi_{\mathrm{onl}}|}{d} &\leq 1 + \frac{2\sum_{i=1}^{2j+1}(1+\delta)\, 2^i}{\frac{1}{1+\delta}\, 2^{2j} - \sum_{i=1}^{2j-1}\left(1+\delta-\frac{1}{1+\delta}\right)\, 2^i + \varepsilon} \\
&= 1 + \frac{2\,(1+\delta)^2\,(4 - \frac{2}{2^{2j}})}{1 - ((1+\delta)^2 - 1)\left(1 - \frac{2}{2^{2j}}\right) + \frac{\varepsilon\,(1+\delta)}{2^{2j}}} \\
&= 1 + \frac{2\,(1+\delta)^2\,(4 - \frac{2}{2^{2j}})}{2 - (1+\delta)^2 + \left(\frac{2}{2^{2j}}((1+\delta)^2 - 1)\right) + \frac{\varepsilon\,(1+\delta)}{2^{2j}}} \\
&< 1 + 8 \cdot \frac{(1+\delta)^2}{2 - (1+\delta)^2}\,,
\end{aligned}
$$

and we achieve a worst case ratio of $1 + 8\, \frac{(1+\delta)^2}{2-(1+\delta)^2}$. For the denominator $2 - (1+\delta)^2 \leq 0$ holds iff $\delta > \sqrt{2} - 1$. □

4 The optimal strategy for known error range

One might wonder whether there is a strategy which takes the error δ into account and yields a competitive factor smaller than the worst case factor of the doubling strategy. Intuitively this seems to be impossible, because the doubling strategy is optimal in the error-free case. We are able to show that there is a strategy that achieves a factor of $1 + 8\left(\frac{1+\delta}{1-\delta}\right)^2$. This is smaller than $1 + 8\frac{1+\delta}{1-3\delta}$ for all $\delta < 1$.

Theorem 2 *In the presence of an error up to* δ *in the percentual error model* $[(1-\delta)f, (1+\delta)f]$ *with* $\delta \in [0, 1[$*, there is a strategy that meets every goal and achieves a competitive factor of* $1 + 8\left(\frac{1+\delta}{1-\delta}\right)^2$.

Proof. We are able to design a strategy as in the error-free case. We can assume that a strategy F is given by a sequence of non-negative values[4], $f_1, f_2, f_3, \ldots$, denoting the nominal values required by the strategy, i. e. in the i-th step the strategy wants the robot to move a distance of f_i to a specified direction— to the right, if i is even and to left if i is odd—and to return to the start point with a movement of f_i to the opposite direction. Remark, that every reasonable strategy can be described in this way.

As above, let ℓ_i^+ and ℓ_i^- denote the length of a movement to the right and to the left in the i-th step, respectively. In the proof of Theorem 1 we showed that every online strategy will achieve a worst case ratio of

$$\frac{|\pi_{\text{onl}}|}{d} = 1 + \frac{\sum_{i=1}^{2j+1}(2\ell_i^-)}{\ell_{2j}^+ - \sum_{i=1}^{2j-1}(\ell_i^- - \ell_i^+) + \varepsilon} ,$$

which achieves its maximum if every step towards the door is as short as possible and every step in the opposite direction is as long as possible, i. e. $\ell_i^- = (1+\delta)\, f_i$ and $\ell_i^+ = (1-\delta)\, f_i$. This yields

$$\frac{|\pi_{\text{onl}}|}{d} = 1 + 2(1+\delta)\frac{\sum_{i=1}^{2j+1} f_i}{(1-\delta)f_{2j} - 2\delta\sum_{i=1}^{2j-1} f_i + \varepsilon} \tag{4}$$

For a fixed δ, 1 and $2(1+\delta)$ are constant, and it is sufficient to find a strategy that minimizes

$$G_{(n,\delta)}(F) := \frac{\sum_{i=1}^{n+1} f_i}{(1-\delta)f_n - 2\delta\sum_{i=1}^{n-1} f_i} \quad \text{if } n \geq 1 \tag{5}$$

and $G_{(0,\delta)}(F) := \frac{f_1}{1}$ where F denotes the strategy $f_1, f_2, f_3, \ldots$ and $G_{(0,\delta)}(F)$ refers to the worst case after the first iteration step. Note, that we assumed that the goal is at least one step away from the start[5].

By simple analysis we found out that a strategy $f_i = \alpha^i$ asymptotically minimizes $G_{(n,\delta)}(F)$ if $\alpha = 2\,\frac{1+\delta}{1-\delta}$ holds. In this case we have

$$\begin{aligned} G_{(n,\delta)}(F) &= \frac{\sum_{i=1}^{n+1} f_i}{(1-\delta)f_n - 2\delta\sum_{i=1}^{n-1} f_i} \\ &= 4\frac{(1+\delta)\left(2\,\frac{1+\delta}{1-\delta}\right)^{n+1} - \frac{1-\delta}{2}}{(1-\delta)^2\left(2\,\frac{1+\delta}{1-\delta}\right)^{n+1} + 4\delta(1-\delta)} < 4\frac{1+\delta}{(1-\delta)^2} \end{aligned} \tag{6}$$

which altogether gives $1 + 8\left(\frac{1+\delta}{1-\delta}\right)^2$ for the competitive ratio. Note, that $f_i = \left(2\,\frac{1+\delta}{1-\delta}\right)^i$ is a reasonable strategy, although handicapped by the error

[4]In the following we use f_i instead of $f(i)$ to omit parenthesis in the equations.

[5]Alternatively, we can assume that the cost in the start situation is subsumed by an additive constant in the definition of the competitive factor.

it monotonically increases the distance to the start which is never smaller than the denominator in (4)

$$(1-\delta)f_n - 2\delta\sum_{i=1}^{n-1} f_i = \frac{1}{(1+3\delta)}\left((1-\delta)(1+\delta)\left(2\,\frac{1+\delta}{1-\delta}\right)^n + 4\delta(1+\delta)\right).$$

Thus, every goal point will be reached. □

We want to show that the given factor is optimal. As we have seen in the proof of Theorem 2 there is a constant upper bound of $G_{(n,\delta)}(F)$ that depends on δ. Now $C(F,\delta) := \sup_n G_{(n,\delta)}(F)$ defines the competitive value of a strategy F, and for every fixed δ there will be a strategy F^* that yields $C^*_\delta := \inf_F C(F,\delta)$. In the following we want to show that there is always a strategy F^* that achieves C^*_δ exactly in every step, that is $G_{(n,\delta)}(F^*) = C^*_\delta$ for $n \geq 1$. For example, for $\delta = 0$ the strategy $f_i = (i+1)2^i$ exactly yields the factor 9 in every step. Note, that up to now only the existence of C^*_δ is known. The idea of using equality was mentioned in [19] and used for a finite strategy in [21]. We give a formal proof for infinite strategies with errors.

Lemma 3 *In the presence of an error up to δ in the percentual error model $[(1-\delta)f, (1+\delta)f]$ with $\delta \in [0,1[$, there is always an optimal strategy F^* that achieves the optimal (sub)factor C^*_δ exactly for all $G_{(n,\delta)}(F^*)$, that is $G_{(n,\delta)}(F^*) = C^*_\delta$ for $n \geq 1$.*

Proof. Let F be a C^*_δ competitive strategy. We can assume that F is strictly positive. We will show that for all $n \geq 1$ there is always a strategy F' that fulfills $G_{(k,\delta)}(F') = C^*_\delta$ for all $1 \leq k \leq n$, by successively adjusting F adequately.

First, note that $G_{(n,\delta)}(F)$ is decreasing in f_n. The value $(1-\delta)f_n - 2\delta\sum_{i=1}^{n-1} f_i$ describes the distance to the start. We can assume that $(1-\delta)f_n - 2\delta\sum_{i=1}^{n-1} f_i > 0$ holds, otherwise the strategy does not increase the distance to the start in this step and there is a better strategy for the worst-case. Now $G_{(n,\delta)}(F)$ equals a function $\frac{f_n+A}{C\cdot f_n - B}$ in f_n with $A, B, C > 0$ and $Cf_n - B > 0$. The function $\frac{f_n+A}{C\cdot f_n - B}$ is positive and strictly decreasing for $Cf_n - B > 0$. The other way round, the function goes to infinity if f_n decreases towards $f_n = \frac{B}{C}$. Altogether, if we decrease f_n adequately we can increase $G_{(n,\delta)}(F)$ continously to whatever we want.

Additionally, $G_{(k,\delta)}(F)$ is increasing in f_n for all $k \neq n$. For $k < n-1$ $G_{(k,\delta)}(F)$ is not affected by f_n. For $k = n-1$ the distance f_n appears only in the nominator of $G_{(k,\delta)}(F)$ which increases if f_n grows. For $k > n$ if f_n grows, the denominator shrinks (it has the coefficient -2δ) and the nominator increases. The other way round, if f_n decreases, $G_{(k,\delta)}(F)$ is decreasing in f_n for all $k \neq n$. Altogether, if we decrease f_n we will decrease all $G_{(k,\delta)}(F)$ for $k \neq n$.

As indicated above let F be a strictly positive C^*_δ competitive strategy. By induction over $n \geq 1$, we will show that we can decrease F to a strictly positive strategy F' which fulfills $G_{(k,\delta)}(F') = C^*_\delta$ for all $1 \leq k \leq n$. Additionally, F' equals F for all f_l and $l \geq n+1$.

For $n = 1$ let us assume that $G_{(1,\delta)}(F) <= C^*_\delta$ holds, otherwise we are done. By the argumentation above we will decrease f_1 by a small value to $f'_1 = f_1 - \varepsilon$ such that $G_{(1,\delta)}(F) = C^*_\delta$ holds. All other $G_{(k,\delta)}(F)$ with $k \neq 1$ will decrease, and the new strategy is still C^*_δ competitive. Since f_2 is strictly positive, f'_1 has to be strictly positive.

For the induction step, we assume that we decrease F to a strictly positive strategy F' such that $G_{(k,\delta)}(F') = C^*_\delta$ for all $1 \leq k \leq n$. F' equals F for all f_l and $l \geq n+1$. Now let $G_{(n+1,\delta)}(F') < C^*_\delta$, otherwise we are done. By the argumentation above we can decrease f_{n+1} by ε to $f'_{n+1} = f_{n+1} - \varepsilon$ such that $G_{(n+1,\delta)}(F') = C^*_\delta$ holds. With the considerations above we know that $G_{(k,\delta)}(F')$ decreases for $k \neq n$. Since f_{n+2} is strictly positive, f'_{n+1} has to be strictly positive.

Unfortunately, at least $G_{(n,\delta)}(F') < C^*_\delta$ holds and we have to apply the induction hypothesis again. We decrease F' to F'' such that $G_{(k,\delta)}(F'') = C^*_\delta$ holds for all $1 \leq k \leq n$. Again F'' is strictly positive. Now in turn we will have $G_{(n+1,\delta)}(F'') < C^*_\delta$ and we start the procedure again by decreasing f'_{n+1} adequately. For convenience we denote F'' by F' again.

Altogether, for every f'_l with $1 \leq l \leq n+1$ in the above process we will have a strictly decreasing sequence which will not decrease towards zero. Therefore every f'_l runs toward a unique positive limit. For the limit values we will have $G_{(k,\delta)}(F') = C^*_\delta$ for $1 \leq k \leq n+1$ which finishes the induction and the proof. $\square$

Fortunately, we can build a recurrence for the optimal strategy F^* with $G_{(n,\delta)}(F^*) = C^*_\delta$ for $n \geq 3$. From $G_{(n-1,\delta)}(F^*) = G_{(n-2,\delta)}(F^*) = C^*_\delta$ we conclude $\sum_{i=1}^{n} f^*_i = C^*_\delta \left((1-\delta)f^*_{n-1} - 2\delta \sum_{i=1}^{n-2} f^*_i\right)$ and $\sum_{i=1}^{n-1} f^*_i = C^*_\delta \left((1-\delta)f^*_{n-2} - 2\delta \sum_{i=1}^{n-3} f^*_i\right)$. Subtracting both yields for all $n \geq 3$

$$f^*_n = C^*_\delta(1-\delta)f^*_{n-1} - C^*_\delta(1+\delta)f^*_{n-2} \qquad (7)$$

Theorem 4 *In the presence of an error up to δ in the percentual error model $[(1-\delta)f, (1+\delta)f]$ with $\delta \in [0,1[$, there is no competitive strategy for searching a point on a line that yields a factor smaller than $1 + 8\left(\frac{1+\delta}{1-\delta}\right)^2$.*

Proof. We solve the recurrence Equation 7 using methods described in Graham et. al. [9]. The characteristic polynom of recurrence Equation 7 is given by

$$X^2 - C^*_\delta(1-\delta)X + C^*_\delta(1+\delta)\,, \qquad (8)$$

which has the roots

$$\lambda, \overline{\lambda} = \frac{1}{2}\left((1-\delta)C_\delta^* \pm \sqrt{C_\delta^*\left(C_\delta^*(1-\delta)^2 - 4(1+\delta)\right)}\right) \tag{9}$$

where $\overline{\lambda}$ denotes the conjugate of λ. Now, the recurrence is given in the closed form $f_n^* = a\lambda^n + \overline{a}\overline{\lambda}^n = 2\mathrm{Re}(a\lambda^n)$ where $\mathrm{Re}(w)$ denotes the real part, c, of a complex number $w = c + di$ and a and $\overline{a}$ are determined by the equations $a + \overline{a} = f_1$ and $a\lambda + \overline{a}\overline{\lambda} = f_2$, and in turn by the starting values f_1 and f_2. If we represent complex numbers by points in the plane, multiplication of two numbers entails adding up the corresponding angles they form with the positive X-axis. If the radiant $C_\delta^*\left(C_\delta^*(1-\delta)^2 - 4(1+\delta)\right)$ of λ is negative, λ is not real and its angle is not equal to 0. Consequently, in this case there exists a smallest natural number s such that $a\lambda^s$ lies in the left halfplane $\{X < 0\}$, so that f_s becomes negative. The roots of the radiant are 0 and $4\frac{(1+\delta)}{(1-\delta)^2}$ which shows that f_n^* gets negative if $C_\delta^* < 4\frac{(1+\delta)}{(1-\delta)^2}$ holds. The optimal strategy F^* with $G_{(n,\delta)}(F^*) = C_\delta^*$ has to be positive, therefore the overall competitive factor has to be at least $1+8\left(\frac{1+\delta}{1-\delta}\right)^2$, which exactly matches the factor of the strategy described in Theorem 2. $\square$

The proof also holds for $\delta = 0$ which gives another proof of the factor 9. Thus, line search with errors is generalized adequately.

Proposition 2 *In the presence of an error up to δ in the standard multiplicative error model $[\frac{1}{(1+\delta)}f, (1+\delta)f]$ for $\delta > 0$ there is a competitive strategy that always meets the goal and achieves a factor of $1 + 8(1+\delta)^4$. There is no strategy that achieves a better competitive factor.*

Proof. We follow the lines of Theorem 2, Lemma 3 and Theorem 4. In the standard multiplicative error model, Equation Equation 4 in the proof of Theorem 2 reads

$$\frac{|\pi_{\mathrm{onl}}|}{d} = 1 + 2(1+\delta)^2 \frac{\sum_{i=1}^{2j+1} f_i}{f_{2j} - \delta(2+\delta)\sum_{i=1}^{2j-1} f_i + \varepsilon}$$

and it suffices to consider the functionals (compare to Equation 5)

$$G_{(n,\delta)}(F) := \frac{\sum_{i=1}^{n+1} f_i}{f_n - \delta(2+\delta)\sum_{i=1}^{n-1} f_i}\,.$$

Analogously, the best doubling strategy $f_i = \alpha^i$ can be found by simple analysis which gives $\alpha = 2(1+\delta)^2$. The corresponding factor $G_{(n,\delta)}(F) < 4\,(1+\delta)^2$ can be computed as shown in (6) which gives an overall factor of $1 + 8(1+\delta)^4$ for the strategy $f_i = (2(1+\delta)^2)^i$.

The strategy proceeds in every iteration step at least by

$$\begin{aligned}
\frac{f_n}{1+\delta} - \Delta_n &= \frac{\alpha^n}{1+\delta} - \frac{\delta\,(2+\delta)}{1+\delta} \cdot \sum_{i=1}^{n-1} \alpha^i \\
&= \frac{\alpha^n}{1+\delta} - \frac{\delta\,(2+\delta)}{1+\delta} \cdot \frac{\alpha^n - \alpha}{\alpha - 1} \\
&= \frac{(1+4\delta+2\delta^2)\,\alpha^n - \delta\,(2+\delta)\,\alpha^n + \delta\,(2+\delta)\,\alpha}{(1+\delta)(1+4\delta+2\delta^2)} \\
&= \frac{(1+2\delta+\delta^2)\,\alpha^n + \delta\,(2+\delta)\,\alpha}{(1+\delta)(1+4\delta+2\delta^2)} > 0 \quad \text{for } \delta > 0
\end{aligned}$$

and the strategy will reach every goal.

It remains to show that the given strategy is optimal. Lemma 3 also holds for the multiplicative error model. With the same techniques we adapt a given strategy such that the optimal factor holds in every step. the corresponding recurrence of an optimal strategy, see Equation 7, is now given by

$$f^*_{n+1} = C^*_\delta f^*_{n-1} - C^*_\delta (1+\delta)^2 f^*_{n-2} \,.$$

We consider the characteristic polynom which is $X^2 - C^*_\delta X + C^*_\delta (1+\delta)^2$, see (8). The polynom has the roots

$$\lambda, \overline{\lambda} = \frac{1}{2}\left(C^*_\delta \pm \sqrt{C^*_\delta (C^*_\delta - 4(1+\delta)^2)} \right)$$

Now with the same arguments as in the proof of Theorem 4 the radiant is non-negative for $C^*_\delta \geq 4(1+\delta)^2$ and the competitive factor has to be at least $1 + 8(1+\delta)^4$. □

5 Error afflicted searching on m rays

The robot is located at the common endpoint of m infinite rays. The target is located on one of the rays, but—as above—the robot neither knows the ray containing the target nor the distance to the target. It was shown by Gal [7] that w.l.o.g. one can visit the rays in a cyclic order and with increasing depth. Strategies with this property are called *periodic* and *monotone*. More precisely, the values f_i of a strategy F denote the depth of a search in the i-th step. Further, f_i and f_{i+m} visit the same ray, and $f_i < f_{i+m}$ holds. An optimal strategy is defined by $f_i = \left(\frac{m}{m-1}\right)^i$.

In the error afflicted setting, the start point of every iteration cannot drift away, since the start point is the only point where all rays meet and the robot has to recognize this point. Otherwise we can not guarantee that all rays are visited. Let us first assume that the error δ is known. Surprisingly, it will turn out that we do not have to distinguish whether δ is known or unknown to the strategy.

Theorem 5 *Assume that an error afflicted robot with error range δ in the percentual error model is given. Searching for a target located on one of m rays using a monotone and periodic strategy is competitive with an optimal factor of*

$$3 + 2\,\frac{1+\delta}{1-\delta}\left(\frac{m^m}{(m-1)^{m-1}} - 1\right)$$

for $\delta < \frac{e-1}{e+1}$.

Proof. A periodic strategy F with nominal values $f_1, f_2, f_3, \ldots$ is monotone if $(1-\delta)f_k > (1+\delta)f_{k-m}$ holds. Now let ℓ_i denote the distance covered by the error afflicted agent in the step i. In analogy to the line case, we achieve the worst case, if the target is slightly missed in step k, but hit in step $k+m$. This yields

$$\frac{|\pi_{\text{onl}}|}{d} = 1 + \frac{2\sum_{i=1}^{k+m-1}\ell_i}{\ell_k + \varepsilon}.$$

This ratio achieves its maximum for F, if we maximize every $\ell_i, i \neq k$ and take a worst case value for ℓ_k. Therefore we set $\ell_k := (1-\beta)f_k$ and $\ell_i := (1+\delta)f_i, i \neq k$ for $\beta \in [-\delta, \delta]$. For convenience we ignore ε from now on. We add $2(1+\delta-(1-\beta))f_k - 2(1+\delta-(1-\beta))f_k) = 0$ to the sum and obtain

$$\begin{aligned} \frac{|\pi_{\text{onl}}|}{d} &= 1 - 2\,\frac{1+\delta-(1-\beta)}{1-\beta} + 2\,\frac{1+\delta}{1-\beta}\,\frac{\sum_{i=1}^{k+m-1} f_i}{f_k} \qquad (10) \\ &= 3 + 2\,\frac{1+\delta}{1-\beta}\left(\frac{\sum_{i=1}^{k+m-1} f_i}{f_k} - 1\right). \end{aligned}$$

The functionals $G_k(F) := \frac{\sum_{i=1}^{k+m-1} f_i}{f_k}$ are identical to the functionals considered in the error-free m-ray search. From these results we know that the strategy $f_i = \left(\frac{m}{m-1}\right)^i$ gives the optimal upper bound $G_k(F) < \frac{m^m}{(m-1)^{m-1}}$, see [3, 7]. Now the adversary has the chance to maximize $3+2\,\frac{1+\delta}{1-\beta}\left(\frac{m^m}{(m-1)^{m-1}} - 1\right)$ over β which obviously gives $\beta = \delta$ Altogether, the factor and the optimality are proven. Note, that the optimal strategy is independent from δ, thus there is no difference between known or unknown error range.

We still have to ensure that $f_i = \left(\frac{m}{m-1}\right)^i$ is monotone which means that the inequality $(1-\delta)\left(\frac{m}{m-1}\right)^k > (1+\delta)\left(\frac{m}{m-1}\right)^{k-m}$ should be fulfilled, which in turn is equivalent to $\delta < \frac{\left(\frac{m}{m-1}\right)^m - 1}{\left(\frac{m}{m-1}\right)^m + 1} =: \delta_{\max}(m)$. Since $\delta_{\max}(m) \longrightarrow_{m\to\infty} \frac{e-1}{e+1} \approx 0.4621$, we know that the best strategy is given by $f_i = \left(\frac{m}{m-1}\right)^i$ if $\delta < 0.4621$ holds. $\square$

Proposition 3 *Assume that an error afflicted robot with error range δ in the standard multiplicative error model is given. Searching for a target located on one of m rays using a monotone and periodic strategy is competitive with an optimal factor of* $3 + 2(1+\delta)^2 \left(\frac{m^m}{(m-1)^{m-1}} - 1\right)$ *for* $\delta < \sqrt{e} - 1$*.*

Proof. With the same arguments as in the proof of Theorem 5 the worst case ratio

$$\frac{|\pi_{\text{onl}}|}{d} = 1 + \frac{2\sum_{i=1}^{k+m-1} \ell_i}{\ell_k + \varepsilon},$$

will be maximized if we set $\ell_k := \frac{1}{(1+\beta)} f_k$ and $\ell_i := (1+\delta) f_i, i \neq k$ for $\beta \geq 0$. We add $2\left((1+\delta) - \frac{1}{1+\beta}\right) f_k - 2\left((1+\delta) - \frac{1}{1+\beta}\right) f_k$ to the sum and achieve

$$\begin{aligned}
\frac{|\pi_{\text{onl}}|}{d} &= 1 - 2(1+\beta)\left((1+\delta) - \frac{1}{1+\beta}\right) + 2(1+\beta)(1+\delta)\frac{\sum_{i=1}^{k+m-1} f_i}{f_k} \\
&= 3 + 2(1+\beta)(1+\delta)\left(\frac{\sum_{i=1}^{k+m-1} f_i}{f_k} - 1\right).
\end{aligned}$$

Analogously, the best strategy for the functional $G_k(F) := \frac{\sum_{i=1}^{k+m-1} f_i}{f_k}$ is given by $f_i = \left(\frac{m}{m-1}\right)^i$ and the adversary has the chance to maximize

$$3 + 2(1+\beta)(1+\delta)\left(\frac{m^m}{(m-1)^{m-1}} - 1\right)$$

which gives $\beta = \delta$.

Preserving for monotonicity means $\frac{1}{(1+\delta)}\left(\frac{m}{m-1}\right)^k > (1+\delta)\left(\frac{m}{m-1}\right)^{k-m}$ should be fulfilled, which in turn is equivalent to $\delta < \sqrt{\left(\frac{m}{m-1}\right)^m} - 1 \leq \sqrt{e} - 1$. □

6 Summary

We have analyzed the standard doubling strategy for reaching a door along a wall in the presence of errors in movements. We showed that the robot is still able to reach the door if the error δ is not greater than $\frac{1}{3}$ (33 per cent on a single step). The competitive ratio of the doubling strategy is given by $8\frac{1+\delta}{1-3\delta} + 1$. The error bound is rather big, so it can be expected that real robots will meet this error. If the maximal error is known in advance the strategy $f_i = \left(2\,\frac{1+\delta}{1-\delta}\right)^i$ is the optimal competitive strategy with a competitive factor of $1 + 8\left(\frac{1+\delta}{1-\delta}\right)^2$. It was shown that the analysis technique can be applied to different error models.

In case of m rays the problem is easier to solve since the robot detects the start point after each return. If the error δ is not greater than $\delta_{\max}(m) = \frac{\left(\frac{m}{m-1}\right)^m - 1}{\left(\frac{m}{m-1}\right)^m + 1}$, which is less than $\frac{e-1}{e+1} \approx 0.46212$ for all m, the standard m-ray doubling strategy with $f_i = \left(\frac{m}{m-1}\right)^i$ is the optimal periodic and monotone strategy and yields a factor $3 + 2\,\frac{1+\delta}{1-\delta}\left(\frac{m^m}{(m-1)^{m-1}} - 1\right)$.

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APPENDIX

A Finding the optimal strategy in Theorem 2

For a fixed δ, 1 and $2(1+\delta)$ is constant, and it is sufficient to find a strategy that minimizes

$$G_{(n,\delta)}(S) := \frac{\sum_{i=1}^{n+1} f_i}{(1-\delta)\, f_n - 2\delta \sum_{i=1}^{n-1} f_i} \quad \text{for } n \geq 1 \tag{11}$$

and $G_{(0,\delta)}(S) := \frac{f_1}{1}$ where S denotes the strategy $f_1, f_2, f_3, \ldots$. $G_{(0,\delta)}(S)$ is the worst case after the first iteration step.

Now, we are searching for a strategy S_α in the form $f_i = \alpha^i$ with a fixed α, possibly depending on δ, that asymptotically minimizes

$$\begin{aligned} G_{n,\delta)}(S_\alpha) &= \frac{\sum_{i=1}^{n+1} \alpha^i}{(1-\delta)\,\alpha^n - 2\delta \sum_{i=1}^{n-1} \alpha^i} \\ &= \frac{\frac{\alpha^{n+2}-\alpha}{\alpha-1}}{(1-\delta)\,\alpha^n - 2\delta\,\frac{\alpha^n-\alpha}{\alpha-1}} \\ &= \frac{\alpha^2 - \frac{1}{\alpha^{n-1}}}{(\alpha-1)(1-\delta) - 2\delta + \frac{2\delta}{\alpha^{n-1}}} \\ &< \frac{\alpha^2}{(1-\delta)\,\alpha - \delta - 1} \quad =: \; H_\delta(\alpha) \end{aligned}$$

To find a minimum of $H_\delta(\alpha)$ we derivate and find the roots

$$\begin{aligned} H'_\delta(\alpha) &= \frac{2\alpha\,((1-\delta)\,\alpha - \delta - 1) - (1-\delta)\,\alpha^2}{((1-\delta)\,\alpha - \delta - 1)^2} \\ &= \frac{(1-\delta)\,\alpha^2 - 2\,(1+\delta)\,\alpha}{(1-\delta)^2\,\alpha^2 - 2\,(1-\delta^2)\,\alpha + (1+\delta)^2} = 0 \\ &\Leftrightarrow \; (1-\delta)\,\alpha^2 - 2\,(1+\delta)\,\alpha = 0 \\ &\Leftrightarrow \; \alpha = 0 \quad \vee \quad \alpha = \frac{2\,(1+\delta)}{1-\delta} \end{aligned}$$

A strategy with $\alpha = 0$ will not move the robot at all, so $\alpha = 2\,\frac{1+\delta}{1-\delta}$ is the only reasonable root. Note, that the denominator of $H'_\delta(2\,\frac{1+\delta}{1-\delta})$ yields $(1+\delta)^2 \neq 0$ for $\delta \geq 0$. To test whether this α is a maximum or minimum, we use the second derivative. Since we want to evaluate $H''_\delta(\alpha)$ only for the roots of the numerator of $H'_\delta(\alpha)$, we can use a simplified form[6]:

[6] The derivative of a function of type $f(x) = \frac{N(x)}{D(x)}$ is $f'(x) = \frac{D(x)\cdot N'(x) - N(x)\cdot D'(x)}{(D(x))^2}$. If we want to evaluate $f'(x)$ only for the roots of $N(x)$, the derivative simplifies to $f'\big|_{N(x)=0}(x) = \frac{N'(x)}{D(x)}$.

$$H_\delta''\Big|_{N'(x)=0}(\alpha) = \frac{2\,(1-\delta)\,\alpha - 2\,(1+\delta)}{(1+\delta)^2}\ .$$

This yields $\frac{2}{1+\delta} > 0$ for $\alpha = 2\,\frac{1+\delta}{1-\delta}$, so we have found a minimum.